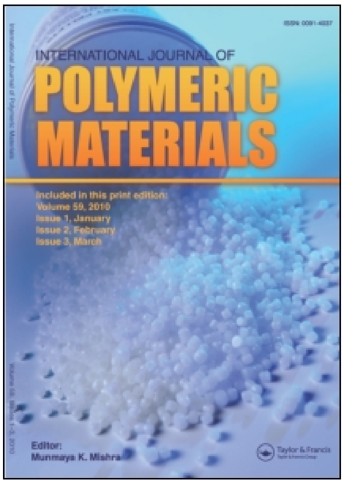

International Journal of Polymeric Materials
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713647664

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Shmuel Brüller[a]
[a] Department of Mechanics, Technion-Israel Institute of Technology, Haifa, Israel

To cite this Article Brüller, Shmuel(1973) 'The Energy Balance of a Viscoelastic Material', International Journal of Polymeric Materials, 2: 2, 137 — 148

To link to this Article: DOI: 10.1080/00914037308075305

URL: http://dx.doi.org/10.1080/00914037308075305

Intern. J. Polymeric Mater., 1973, Vol. 2, pp. 137–148

Printed in Great Britain

The Energy Balance of a Viscoelastic Material

SHMUEL BRÜLLER

Department of Mechanics, Technion—Israel Institute of Technology, Haifa, Israel

(*Received October* 30, 1972)

It is well known that the work done by external forces on a viscoelastic material is converted into a conserved part (potential energy) and a dissipated part, each of which may be divided into two other parts: the isotropic one which is connected with volume changes and the deviatoric one which is associated with distortions.

For strain and strain-rate-independent Poisson's ratio (which is reported to be the case for most viscoelastic materials) the time-dependent isotropic and deviatoric moduli differ only by a constant factor. Expressing the relaxation moduli by Prony–Dirichlet series enables the evaluation of the isotropic and deviatoric parts of the stress-power. These calculations are carried out for the case of constant strain-rate uniaxial tension. The positive definite terms of the resulting expression stand for the dissipated stress-power and the remaining terms—for the conserved stress-power. By integrating over time, the different parts of the stress-energy are obtained.

The ratio of deviatoric part to isotropic part of energy is found to be independent of time and equal for both conserved and dissipated energies.

Results of experiments carried out on Perspex (polymethyl methacrylate) and epoxy-resin were used to calculate the different parts of stress-energy. It is found that the ratio of dissipated energy to conserved energy is always smaller than unity decreasing for smaller strains and strain-rates. The energy computations are practically not affected by the choice of the parameters representing the viscoelastic behaviour of material.

The proposed method can be easily applied to other experimental conditions such as relaxation, creep, constant rate of stress or any other loading history.

LIST OF SYMBOLS

B_i —material constant

E_i —material constant

$E(t)$ —longitudinal relaxation modulus

$G_1(t)$ —deviatoric relaxation modulus

$G_2(t)$	—isotropic relaxation modulus
H	—sum of squared deviations
U	—coefficient matrix of normal equations
W	—stress–energy per unit volume
d	—deviatoric strain
s	—deviatoric stress
t	—time
v_i	—deviation
δ_{ij}	—Kronecker's delta
ϵ	—strain
ϵ^*	—isotropic strain
ν	—Poisson's ratio
σ	—stress
σ^*	—isotropic stress
τ	—time
τ_i	—relaxation time.

INTRODUCTION

During loading of a viscoelastic material, the work done by the external forces is converted into a conserved part (potential energy) and a dissipated part, each of which may be divided into two other parts: the isotropic one which is connected with volume changes and the deviatoric one which is associated with distortions.

If we denote by $\sigma_{ij}(t)$, $\epsilon_{ij}(t)$, $s_{ij}(t)$, $d_{ij}(t)$ the components of the tensors of stress, strain, stress deviator and strain deviator respectively at the time t, the following relations hold:

$$\begin{gathered} s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{\alpha\alpha} \\ d_{ij} = \epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\epsilon_{\alpha\alpha} \\ (i, j, \alpha = 1, 2, 3) \end{gathered} \tag{1}$$

The isotropic parts of the stress and strain tensors denoted respectively by σ_{ij}^* and ϵ_{ij}^* are given by:

$$\begin{gathered} \sigma_{ij}^* = \tfrac{1}{3}\delta_{ij}\sigma_{\alpha\alpha} \\ \epsilon_{ij}^* = \tfrac{1}{3}\delta_{ij}\epsilon_{\alpha\alpha} \end{gathered} \tag{2}$$

(Repeated Greek index denotes summation and δ_{ij} is Kronecker's delta.)

The integral operator form of the constitutive relations based on relaxation modulus is adopted (cf. Ref. 1 for example):

$$s_{ij}(t) = 2\int_0^t G_1(t-\tau)\frac{\partial d_{ij}}{\partial \tau}d\tau \tag{3}$$

$$\sigma_{aa}^*(t) = 2\int_0^t G_2(t-\tau)\frac{\partial \epsilon_{aa}^*}{\partial \tau}d\tau \tag{3'}$$

The functions $G_1(t)$ and $G_2(t)$ represent the deviatoric and isotropic relaxation moduli respectively, at uniform temperature.

In simple tension with constant strain-rate $\dot{\epsilon}$ (which is the case for most experimental devices) the non-vanishing deviatoric components of the stress and strain tensors are (ν denotes Poisson's ratio):

$$\begin{aligned} s_{11} &= \tfrac{2}{3}\sigma_{11} & s_{22} &= s_{33} = -\tfrac{1}{3}\sigma_{11} \\ d_{11} &= \tfrac{2}{3}\dot{\epsilon}t(1+\nu) & d_{22} &= d_{33} = -\tfrac{1}{3}\dot{\epsilon}t(1+\nu) \end{aligned} \tag{4}$$

The isotropic components are in this case:

$$\begin{aligned} \sigma_{11}^* &= \sigma_{22}^* = \sigma_{33}^* = \tfrac{1}{3}\sigma_{11} \\ \epsilon_{11}^* &= \epsilon_{22}^* = \epsilon_{33}^* = \tfrac{1}{3}\dot{\epsilon}t(1-2\nu) \end{aligned} \tag{5}$$

Tensile experiments carried out by the author[2] on two different viscoelastic materials, Perspex and epoxy-resin show that under normal isothermal engineering conditions Poisson's ratio may be considered constant in a relatively large range of strain and strain-rate values (see also Rigbi's survey[3]).

For constant Poisson's ratio the relaxation law Eqs. (3) and (3′) leads to:

—for the deviatoric components:

$$\sigma_{11} = 2\dot{\epsilon}(1+\nu)\int_0^t G_1(t-\tau)d\tau \tag{6}$$

—for the isotropic components:

$$\sigma_{11} = 2\dot{\epsilon}(1-2\nu)\int_0^t G_2(t-\tau)d\tau \tag{6'}$$

It is easy to show that:

$$G_2(t) = \frac{1+\nu}{1-2\nu}G_1(t) \tag{7}$$

(Eq. (7) can be obtained from the general relation presented in Ref. 1 putting $\nu = \text{const.}$)

In addition to $G_1(t)$ and $G_2(t)$ the longitudinal relaxation modulus $E(t)$ can be defined from:

$$\sigma_{11} = 2\int_0^t E(t - \tau)\frac{\partial\epsilon_{11}}{\partial\tau}d\tau \tag{8}$$

The following relations between $E(t)$ and $G_1(t)$ and $G_2(t)$ are obtained:

$$E(t) = (1 + \nu)G_1(t) \tag{9}$$

$$E(t) = (1 - 2\nu)G_2(t) \tag{10}$$

ENERGY BALANCE

a Constant strain-rate

For uniaxial tension with constant strain-rate the total stress-power per unit volume is given by:

$$\dot{W} = \sigma_{11}\dot{\epsilon}_{11} = s_{11}\dot{d}_{11} + 2s_{22}\dot{d}_{22} + 3\sigma_{11}{}^*\dot{\epsilon}_{11}{}^* \tag{11}$$

Substitution from Eqs. (4) and (5) leads to:

$$\dot{W} = \tfrac{2}{3}\sigma_{11}\dot{\epsilon}_{11}(1 + \nu) + \tfrac{1}{3}\sigma_{11}\dot{\epsilon}_{11}(1 - 2\nu) \tag{12}$$

where the first term represents the deviatoric stress-power $\dot{W}_{dr}$ and the second term the isotropic stress-power $\dot{W}_i$. If we omit tensor index for simpler writing Eq. (12) will be:

$$\dot{W} = \tfrac{2}{3}\sigma\dot{\epsilon}(1 + \nu) + \tfrac{1}{3}\sigma\dot{\epsilon}(1 - 2\nu) \tag{12'}$$

Now, we express the relaxation modulus in the well-known form of a Dirichlet–Prony series:

$$E(t) = E_0 + E_a e^{-t/\tau_a} \tag{13}$$

where E_0 and E_a are $m + 1$ real constants and the m real relaxation times τ_a are non-negative.

We are led to the following stress–strain relation:

$$\sigma = 2\dot{\epsilon}[E_0 t + E_a\tau_a(1 - e^{-t/\tau_a})] = \sigma_0 + \sum_{i=1}^{m}\sigma_i \tag{14}$$

where:

$$\sigma_0 = 2E_0\dot{\epsilon}t = 2E_0\epsilon \tag{15}$$

$$\sigma_i = 2\dot{\epsilon}E_i\tau_i(1 - e^{-t/\tau_i}) \tag{16}$$

The total stress-power will be:

$$\dot{W} = \dot{\epsilon}\left(\sigma_0 + \sum_{i=1}^{m}\sigma_i\right) = \sigma_0\dot{\epsilon} + \dot{\epsilon}\sum_{i=1}^{m}\sigma_i \tag{17}$$

σ_i as given by Eq. (16) is the solution of the differential equation (cf. Ref. 4):

$$\frac{\sigma_i}{2E_i\tau_i} + \frac{\dot{\sigma}_i}{2E_i} = \dot{\epsilon} \tag{18}$$

Substitution of σ_0 from Eq. (15) and $\dot{\epsilon}$ from Eq. (18) into Eq. (17) yields:

$$\dot{W} = 2E_0\epsilon\dot{\epsilon} + \frac{1}{2E_a}\sigma_a\dot{\sigma}_a + \frac{1}{2E_a\tau_a}\sigma_a\sigma_a \tag{19}$$

In Eq. (19) the terms of the form $\sigma_a\sigma_a$ are independent of direction and therefore represent the dissipated stress-power, while the remaining terms represent the conserved stress-power.

Integrating Eq. (19) we obtain the dissipated and conserved stress-energy and according to Eq. (12′) the four components of the total stress-energy are:

i) The dissipated deviatoric stress-energy:

$$W_{dd} = \tfrac{2}{3}(1 + \nu)\dot{\epsilon}^2\{2tE_a\tau_a + E_a\tau_a{}^2[1 - (2 - e^{-t/\tau_a})^2]\} \tag{20}$$

ii) The conserved deviatoric stress-energy:

$$W_{cd} = \tfrac{2}{3}(1 + \nu)\dot{\epsilon}^2[E_0t^2 + E_a\tau_a{}^2(1 - e^{-t/\tau_a})^2] \tag{21}$$

iii) The dissipated isotropic stress-energy:

$$W_{di} = \tfrac{1}{3}(1 - 2\nu)\dot{\epsilon}^2\{2tE_a\tau_a + E_a\tau_a{}^2[1 - (2 - e^{-t/\tau_a})^2]\} \tag{22}$$

iv) The conserved isotropic stress-energy:

$$W_{ci} = \tfrac{1}{3}(1 - 2\nu)\dot{\epsilon}^2[E_0t^2 + E_a\tau_a{}^2(1 - e^{-t/\tau_a})^2] \tag{23}$$

It should be mentioned that the rate of energy dissipation is given by the last term of Eq. (19):

$$\dot{W}_d = \frac{1}{2E_a\tau_a}\sigma_a\sigma_a = 2\dot{\epsilon}^2E_a\tau_a(1 - e^{-t/\tau_a})^2 \tag{24}$$

and its deviatoric and isotropic parts may be evaluated as shown above.

b Stress relaxation

If at the time $t = 0$ the specimen is suddenly stretched until the strain reaches the value $\epsilon_{(0)}$, the work done by the external forces per unit volume is:

$$W = \tfrac{1}{2}\sigma_{(0)}\epsilon_{(0)} \tag{25}$$

where the stress $\sigma_{(0)}$ may be calculated from:

$$\sigma = 2\epsilon_{(0)}E(t) = 2\epsilon_{(0)}(E_0 + E_a e^{-t/\tau_a}) \tag{26}$$

Putting $t = 0$, i.e.:

$$\sigma_{(0)} = 2\epsilon_{(0)}\left(E_0 + \sum_{a=1}^{m} E_a\right) \tag{27}$$

the work done by the external forces is:

$$W = \epsilon^2_{(0)}\left(E_0 + \sum_{a=1}^{m} E_a\right) \tag{28}$$

At time t the conserved energy is:

$$W_c = \tfrac{1}{2}\sigma\epsilon_{(0)} = \epsilon^2_{(0)}(E_0 + E_a e^{-t/\tau_a}) \tag{29}$$

The dissipated energy will be the difference between the work done by the external forces and the conserved energy:

$$W_d = W - W_c = \epsilon^2_{(0)}(1 - e^{-t/\tau_a})E_a \tag{30}$$

As in the previous case the deviatoric and the isotropic parts of these energies may be calculated, multiplying them by the constant factors $\tfrac{2}{3}(1 + \nu)$ and $\tfrac{1}{3}(1 - 2\nu)$ respectively.

The rate of energy dissipation is:

$$\dot{W}_d = \epsilon^2_{(0)}\frac{E_a}{\tau_a}e^{-t/\tau_a} \tag{31}$$

c Constant rate of stress and creep

Starting from the integral operator form of the constitutive equation based on creep compliance the method presented above can easily be adapted to compute the different parts of the stress energy for the case of constant rate of stress or creep.

EXPERIMENTAL RESULTS

Tensile tests were carried out on two materials, Perspex (polymethyl methacrylate) and epoxy-resin (Versamid 140—40% and Epikote 815—60%) at four constant strain-rates with differences of about one decade. The experimental set-up consisted of a 5-ton Instron Testing Machine, a strain-gage bridge and an additional recorder. Axial strain and load were recorded for each individual test.

Another type of experiments was carried out for the determination of Poisson's ratio. The results of these experiments show that Poisson's ratio is practically constant and independent of the strain-rate for both materials, its values being:

$$\nu = 0.395 \text{ for Perspex}$$

$$\nu = 0.400 \text{ for epoxy-resin}$$

The collocation method discussed in the Appendix was used for fitting the viscoelastic stress response of the materials to Eq. (14). It was found that for each constant strain-rate two time-dependent terms of series (Eq. (13)) in addition to the constant term lead to excellent fits. The five constants obtained for each strain-rate enable the calculation of the different parts of the stress-energy as given by Eqs. (20)–(23). The results, plotted versus strain, are shown in Figures 1–6.

DISCUSSION

According to Eqs. (20)–(23), for constant Poisson's ratio, the deviatoric and isotropic parts are constant ratios of the stress-energy.

Taking $\nu = 0.4$ the deviatoric part represents 93.3% of the stress-energy while the isotropic part represents only 6.7%.

The total stress energy as well as the deviatoric and isotropic parts of it are depicted in Figure 1 for Perspex and in Figure 4 for epoxy-resin. It is seen that each of these energies is larger for higher strain-rates and obviously increases with the strain.

For Perspex, the same conclusions may be drawn for the conserved strain-energy as shown in Figure 2 (i.e. larger energy corresponds to higher strain-rates and energy increases with strain), whereas the dissipated stress-energy is smaller for higher strain-rates but continues to increase with the strain (see Figure 3).

The behavior of epoxy-resin has been found to be different. For the two

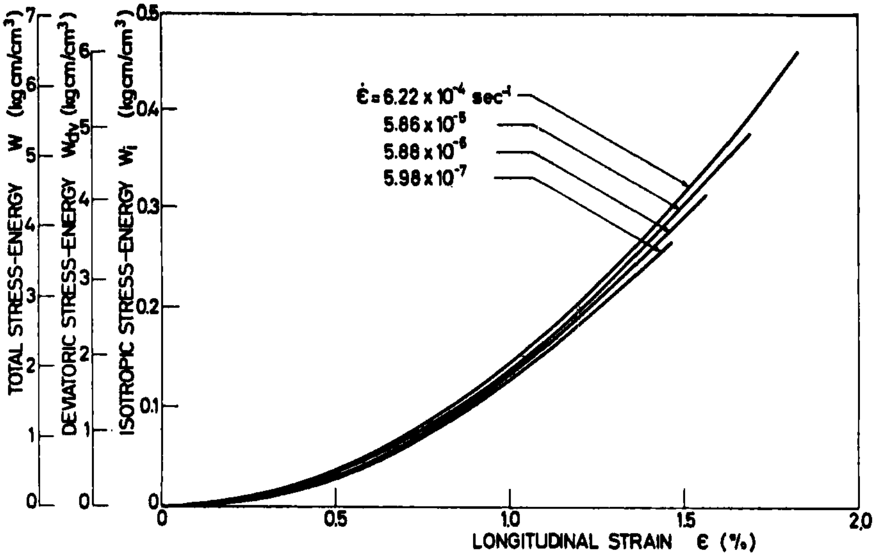

FIGURE 1 Stress-energy of Perspex.

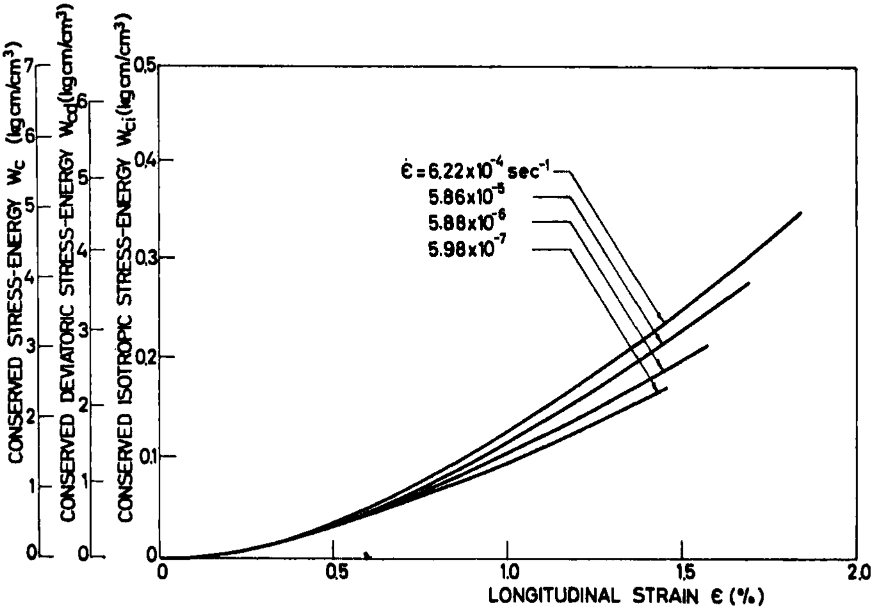

FIGURE 2 Conserved stress-energy of Perspex.

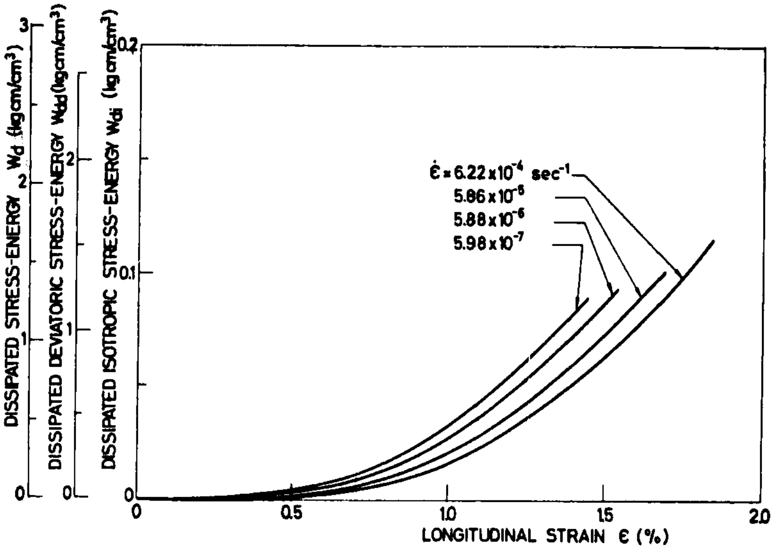

FIGURE 3 Dissipated stress-energy of Perspex.

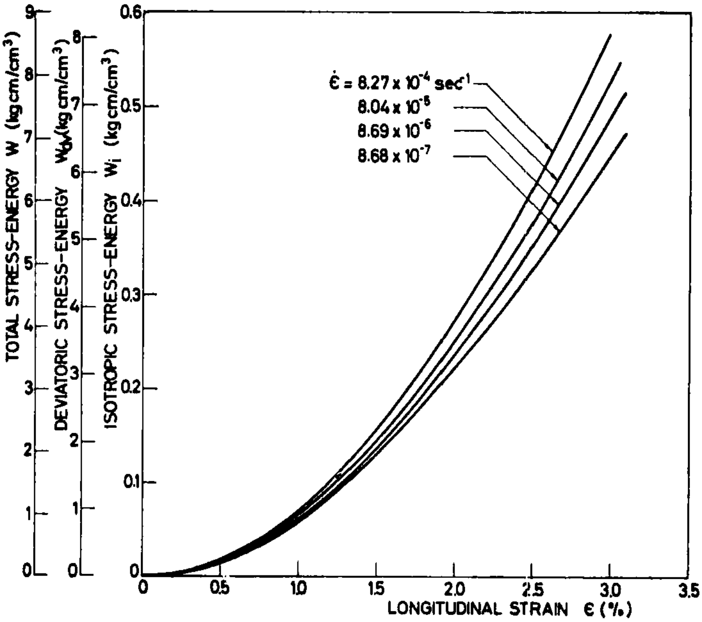

FIGURE 4 Stress-energy of epoxy-resin.

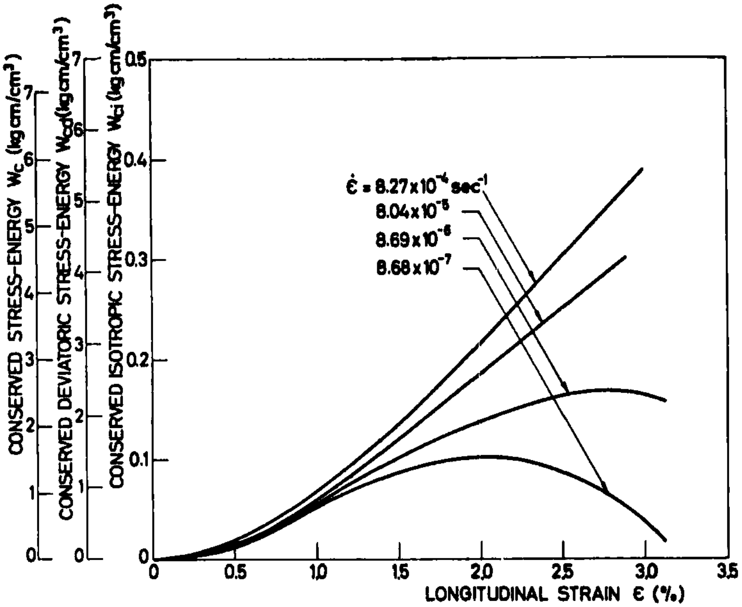

FIGURE 5 Conserved stress-energy of epoxy-resin.

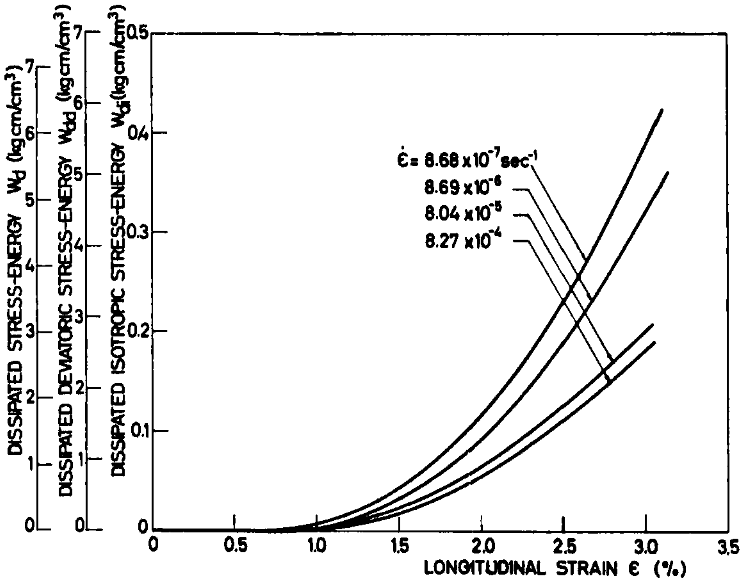

FIGURE 6 Dissipated stress-energy of epoxy-resin.

higher strain-rates which were examined, the conserved stress-energy increases with strain—while for the two lower strain-rates, the conserved energy increases until the strain reaches a certain value and then decreases as shown in Figure 5. For the two lower strain-rates the dissipated stress-energy of epoxy-resin increases quickly and reaches relatively high values. This phenomenon can be explained by the fact that from a certain value of strain the rate of dissipation of energy is larger than the rate at which energy is produced by the external forces.

The above-presented method can be used to compute the different parts of the stress-energy for constant rate of stress, stress-relaxation or creep. Moreover, the method can be extended for more complicated loading histories.

Finally it should be mentioned that the choice of the $2m + 1$ constants governing the viscoelastic behavior of the material does not practically affect the results obtained.

APPENDIX

Determination of the parameters describing the viscoelastic behavior of material

The problem is to determine the $2m + 1$ constants ($E_0, E_1, \ldots, E_m, \tau_1, \ldots, \tau_m$) which describe the material behavior. Let us consider the case of constant strain-rate experiments where the stress is given by Eq. (14):

$$\sigma = 2\dot{\epsilon}[E_0 t + E_a \tau_a (1 - e^{-t/\tau_a})]$$

or:

$$\sigma' = E_0 t + B_a (1 - e^{-t/\tau_a}) \tag{a}$$

where:

$$\sigma' = \frac{\sigma}{2\dot{\epsilon}} \tag{b}$$

$$B_i = E_i \tau_i \tag{c}$$

Using a method as presented in Ref. 5 we shall choose the m relaxation times τ_a in the range of the time-scale of the considered experiment.

The remaining $m + 1$ constants will be obtained by using least-squares techniques.

If the experimental values are $\sigma_1', \sigma_2', \ldots, \sigma_n'$ ($n > m + 1$), expressing the n deviations ν_i of the experimental values σ_i' from the values calculated at t_i, we obtain the system:

$$\nu_i = E_0 t_i + B_a (1 - e^{-t_i/\tau_a}) - \sigma_i' \tag{d}$$

We define the sum:

$$H = \nu_a \nu_a \tag{e}$$

H will be minimal if the following $m + 1$ conditions are satisfied:

$$\frac{\partial H}{\partial E_0} = 0; \quad \frac{\partial H}{\partial B_a} = 0 \tag{f}$$

$$(a = 1, 2, \ldots m)$$

Conditions (f) lead to a system of normal equations:

$$U^T U B = U^T \sigma' \tag{g}$$

U is the coefficient matrix of $(m + 1) \times n$ terms

$$U = \begin{bmatrix} t_1 \ (1 - e^{-t_1/\tau_1}) \ldots\ldots\ldots\ldots (1 - e^{-t_1/\tau_m}) \\ t_2 \ (1 - e^{-t_2/\tau_1}) \ldots\ldots\ldots\ldots (1 - e^{-t_2/\tau_m}) \\ . \\ \\ . \\ t_n \ (1 - e^{-t_n/\tau_m}) \ldots\ldots\ldots\ldots (1 - e^{-t_n/\tau_m}) \end{bmatrix} \tag{h}$$

B is the vector of the $m + 1$ unknowns

$$B = \begin{bmatrix} E_0 \\ B_1 \\ . \\ . \\ B_m \end{bmatrix} \tag{i}$$

σ' is the vector of the free terms and U^T is the transpose of matrix U.

The $m + 1$ parameters $E_0, B_1, \ldots B_m$ can now be calculated from the system of normal equations (g).

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